

Probing Fractionalized Charges

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Abstract

Inspired by the holographic entanglement entropy, for geometries with non-zero abelian charges, we define a quantity which is sensitive to the background charges. One observes that there is a critical charge below that the system is mainly described by the metric and the effects of the background charges are just via metric's components. While for charges above the critical one the background gauge field plays an essential role. This, in turn, might be used to define an order parameter to probe phases of a system with fractionalized charges.

1 Introduction

In application of AdS/CFT correspondence [1] to condensed matter physics, one typically is interested in a gravity dual which describes a system at finite temperature and density. Following [2] a natural guess for the dual gravity would be a charged black hole. The existence of the charged horizon would result to a dual theory at finite temperature and finite density.

We, note, however that this is not the only way to construct a gravity model whose dual theory is a system at finite density. Indeed finite density holographic duals may be obtained by two, rather distinctive, ways. Actually the asymptotic electric flux-to be identified with the chemical potential at the boundary theory- may be supported by either non-zero charges from behind an event horizon, or charged matter in the bulk geometry. If we are interested in a phase with unbroken $U(1)$ global group, the matter field in the bulk is charged fermions (see for example [3]).

Of course one can distinguish between these two cases due to the fact that in the first case (fractionalized phase) the charge density is of order of N^2 while in the second case (mesonic phase) it is of order $\mathcal{O}(N^0)$, where N is the number of degrees of freedom (the number of color for $U(N)$ gauge theory). Alternatively, when the $U(1)$ is unbroken, the fractionalized phase may also be identified by the violation of the Luttinger theorem [4–6].

Since the charge density of a system may be originated by both from behind an event horizon and a charged matter, it could be in different phases depending on the origin of the asymptotic flux. To classify the possible phases an order parameter has been introduced in [7]. The order parameter at leading order is essentially the holographic entanglement entropy with taking into account the electric flux through the hypersurface of holographic entanglement entropy. In the present paper we would like to introduce an order parameter which may probe the fractionalized phases.

To proceed, let us consider a $d + 2$ dimensional Einstein-Dilaton-Maxwell theory whose action, in minimal form, may be written as follows

$$I = \frac{1}{16\pi G_{d+2}} \int d^{d+2}x \sqrt{-g} \left[R - \frac{1}{2}(\partial\phi)^2 + V(\phi) - \frac{1}{4} \sum_{i=1}^n e^{\lambda_i \phi} F_i^2 \right]. \quad (1.1)$$

where G_{d+2} is the $d + 2$ dimensional Newton constant and λ_i 's are parameters of model. This is, indeed, a typical action we get from compactification of low energy effective action of string theory. Of course this is the case for particular values of the parameters λ_i and a specific form of the potential. Nevertheless in what follows we will not restrict ourselves to these particular values.

A generic solution of the equations of motion of the above action could be a charged black hole (brane) with non-trivial dilaton profile. We may assume that the background solution to be an asymptotically locally AdS_{d+2} . Therefore the solution may provide a gravitational dual for a $d + 1$ dimensional theory at finite charge and

temperature with a UV fixed point.

Consider a closed subsystem of the $d+1$ dimensional dual field theory. Then it is natural to pose a question of what kind of information is encoded in this subsystem and how one can extract this information. To address these questions it might be useful to look at the holographic dual of the theory.

Indeed this is what we are familiar with in the context of AdS/CFT correspondence. Prototype examples include holographic entanglement entropy [8] and Wilson loop [9, 10]. In both cases the gravitational dual is found useful for extracting the corresponding information. In both cases the problem reduces to minimizing an area of a hypersurface in the bulk gravity. Actually motivated by these quantities we would like to define a similar object which is also sensitive to the background gauge field.

We note, however, that since typically we are interested in backgrounds with electric field it is not appropriate to work with fixed time as one does for the holographic entanglement entropy. In other words it would be more natural to consider the geometric entropy [11, 12] which is essentially similar to the entanglement entropy though one considers a hypersurface with a spatial direction fixed. It worth noting that to compute the geometric entropy one usually utilizes a double Wick rotation to promote a spatial direction to time direction, though we note that, as far as the computations are concerned it is not necessary to do that.

Motivated by the above discussions, let us define the following quantity¹

$$\Gamma = \frac{1}{G_{d+2}} \int d^d \xi \sqrt{-\det \left(\tilde{g} + R \sum_i^n F_{ab}^i \right)} \quad (1.2)$$

where ξ_a 's are the coordinates on the hypersurface and R is a typical scale of the theory (*e.g.* the radius of curvature). Moreover²

$$\tilde{g}_{ab} = \frac{\partial x^\mu}{\partial \xi_a} \frac{\partial x^\nu}{\partial \xi_b} g_{\mu\nu}, \quad F_{ab}^i = \frac{\partial x^\mu}{\partial \xi_a} \frac{\partial x^\nu}{\partial \xi_b} F_{\mu\nu}^i, \quad \text{for } x^d = \text{fixed} \quad (1.3)$$

are the induced metric and the pull back of the gauge field on the world volume of the hypersurface, respectively. An advantage of this definition is that, it is directly sensitive to the background charge. This is in contrast to the holographic entanglement entropy or Wilson loop where the effects of the background charge is due to the metric components.

Following the general idea of AdS/CFT correspondence, it is then natural to minimize Γ . The resultant quantity might be used to define an order parameter which could probe different phases of the system as we will demonstrate in the

¹In general one could have put free parameters in front of each F_{ab}^i 's in the square root and therefore one has an n -parameter family object. We would like to thank D. Tong for suggesting such a possibility.

²In what follows we use a notation in which $x^\mu = (t, r, x^i)$ for $i = 1, \dots, d$.

following sections, within a specific model.

The paper is organized as follows. In the next section we will consider charge black branes with one $U(1)$ charge and then compute the quantity (1.2) where we explore its different properties. In section three we redo the same computations for the charge black hole in a global AdS geometry. In section four we will proposed a new quantity which might be thought of as a generalization of the equation (1.2). The last section is devoted to the discussions,

2 Electrically charged black brane solutions

In this section in order to explore a possible information encoded in the expression defined by (1.2) we will consider a particular model consisting of the Einstein gravity with a negative cosmological constant coupled to a $U(1)$ gauge field. In this case the action (1.1) reduces to

$$I = \frac{1}{16\pi G_{d+2}} \int d^{d+2}x \sqrt{-g} \left(R - 2\Lambda - \frac{1}{4}F^2 \right). \quad (2.1)$$

This model admits several vacuum solutions which could be either electric or dyonic black branes (holes) charged under the $U(1)$ gauge field. In what follows we will consider the electric case and will postpone the dyonic one to the section four.

Let us consider a $d+2$ dimensional Reissner-Nordstrom AdS black brane solution which for $d \geq 2$ may be written as follows [13]³

$$ds^2 = \frac{R^2}{r^2} \left(-f(r)dt^2 + \frac{dr^2}{f(r)} + \sum_{i=1}^d dx_i^2 \right), \quad F_{rt} = -QR\sqrt{2d(d-1)}r^{d-2},$$

$$f(r) = 1 - (1 + Q^2 r_H^{2d}) \left(\frac{r}{r_H} \right)^{d+1} + Q^2 r^{2d}, \quad (2.2)$$

where $R = \sqrt{-\frac{d(d+1)}{2\Lambda}}$ and r_H are the radii of curvature and horizon, respectively. The Hawking temperature in terms of the radius of the horizon is

$$T = \frac{d+1}{4\pi r_H} \left(1 - \frac{d-1}{d+1} Q^2 r_H^{2d} \right). \quad (2.3)$$

This geometry is supposed to provide a gravitational description for a $d+1$ dimensional CFT at finite temperature and density. The corresponding chemical potential is

$$\mu = \sqrt{\frac{2d}{d-1}} QR r_H^{d-1}. \quad (2.4)$$

³Actually for $d = 1$ we still have the same solution but with $f = 1 - r^2 + \frac{Q^2}{2}r^2 \ln r$ and $F_{rt} = \frac{Q}{r}$.

Let us consider the following strip as a subsystem in the dual $d + 1$ dimensional theory

$$0 \leq t \leq \tau, \quad -\frac{\ell}{2} \leq x_{d-1} \leq \frac{\ell}{2}, \quad 0 \leq x_i \leq L, \quad x_d = \text{fixed} \quad (2.5)$$

for $i = 1, \dots, d - 2$. Then there is a hypersurface in the bulk whose intersection with the boundary coincides with the above strip. The profile of the corresponding hypersurface may be given by $x_{d-1} = x(r)$. Thus the induced metric on the hypersurface is

$$ds_{\text{ind}}^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = \frac{R^2}{r^2} \left[-f dt^2 + \left(\frac{1}{f} + x'^2 \right) dr^2 + \sum_{i=1}^{d-2} dx_i^2 \right], \quad (2.6)$$

where prime represents derivative with respect to r . In this case the expression (1.2), taking into account the solution (2.2) and the boundary subsystem (2.5), reads

$$\Gamma = \frac{\tau L^{d-2} R^d}{G_{d+2}} \int dr \frac{\sqrt{1 - \phi^2 + f x'^2}}{r^d}, \quad (2.7)$$

where $\phi = \sqrt{2d(d-1)} Q r^d$.

Now the aim is to minimize Γ . Actually there is a standard procedure to minimize Γ by which the expression of Γ may be treated as a one dimensional action for x whose momentum conjugate is a constant of motion. Therefore one arrives at

$$\frac{f x'}{r^d \sqrt{1 - \phi^2 + f x'^2}} = c, \quad (2.8)$$

where c is a constant which can be fixed at a particular point. Usually the particular point is chosen to be the turning point where $x' \rightarrow \infty$ in which x' drops from the left hand side leading to a constant which is given in terms of a function of r evaluated at the turning point. When we are not explicitly considering the effects of gauge field, *e.g.* in the computation of holographic entanglement or geometric entropies where there is no F in the square root, then the position of turning point is located between boundary and horizon. Whereas in the present case the situation is different.

Actually as we shall see when we increase the background charges the effects of gauge field become important leading to a new scale in the theory which could take over the role of the horizon. More precisely, as it is evident from the equation (2.8), for a given background charge there is a special point at which $\phi = 1$ that is given by

$$r_\phi = \left(\frac{1}{2d(d-1)Q^2} \right)^{\frac{1}{2d}}. \quad (2.9)$$

Note that although at this point the x' dependence is dropped from the left hand side of the equation (2.8), it is not a turning point. Moreover one can convince ourselves that the minimization makes sense only for $r \leq r_\phi$. In other words, in the

present case the location of the turning point will be between boundary and r_{\min} where $r_{\min} = \text{Min}(r_H, r_\phi)$, *i.e.* $0 \leq r_t \leq r_{\min}$, with r_t being the turning point. In what follows we will consider both $r_{\min} = r_H$ and $r_{\min} = r_\phi$ cases.

2.1 $r_{\min} = r_H$ case

Let us assume $r_{\min} = r_H$ which happens if

$$Q \leq Q_c = \frac{1}{\sqrt{2d(d-1)r_H^d}}, \quad \text{or} \quad \mu \leq \mu_c = \frac{R}{(d-1)r_H}. \quad (2.10)$$

In this case one finds

$$\ell = 2 \int_0^{r_t} dr \left(\frac{f_t}{f^2} \right)^{1/2} \left(\frac{r}{r_t} \right)^d \frac{\sqrt{1-\phi^2}}{\sqrt{1 - \left(\frac{r}{r_t} \right)^{2d} \frac{f_t}{f}}}. \quad (2.11)$$

where $f_t = f(r_t)$. On the other hand using the equation (2.8) one arrives at

$$\Gamma = \frac{\tau L^{d-2} R^d}{G_{d+2}} \int_\epsilon^{r_t} dr \frac{\sqrt{1-\phi^2}}{r^d \sqrt{1 - \left(\frac{r}{r_t} \right)^{2d} \frac{f_t}{f}}}, \quad (2.12)$$

where ϵ is a UV cut off. From these expressions it is clear that there is a new scale in the theory that controls the effects of the background field, as we anticipated. Of course since for the moment we are in the range of $Q \leq Q_c$, the new scale is irrelevant in what follows. We will back to the case of $Q > Q_c$ latter.

If one drops the factor of $\sqrt{1-\phi^2}$, the above expressions reduce to that of the geometric entropy studied in [11, 12]. Moreover for pure AdS_{d+2} , $d \geq 2$ one has [8]

$$\Gamma = \frac{\tau L^{d-2} R^d}{G_{d+2}} \left[\frac{1}{(d-1)\epsilon^{d-1}} - \frac{2^{d-1}\pi^{d/2}}{d-1} \left(\frac{\Gamma\left(\frac{d+1}{2d}\right)}{\Gamma\left(\frac{1}{2d}\right)} \right)^d \frac{1}{\ell^{d-1}} \right], \quad (2.13)$$

which is the expression of holographic entanglement entropy. Note also that for $d = 1$ one gets a logarithmic behavior, $\Gamma \sim \ln \frac{\ell}{\epsilon}$.

For the RN background given in the equation (2.2) we cannot find an analytic expression for Γ as a function of ℓ . Nevertheless we can utilize a numerical method to find $\Gamma(\ell)$ numerically. This is, indeed, what we shall do in this subsection. To proceed let us first explore the behavior of ℓ as a function of r_t .

From the expression (2.11) one finds that for sufficiently small r_t where Γ probes the UV region of the theory the width ℓ vanishes as $\ell \sim r_t \rightarrow 0$. Moreover, in the opposite limit, the width ℓ also goes to zero as the turning point approaches the horizon. It is, indeed, due to the facts that $f_t \rightarrow 0$ for $r_t \rightarrow r_H$ and also the integrand does not diverge faster than $1/f_t$. Therefore for $0 \leq r_t \leq r_H$ the width ℓ

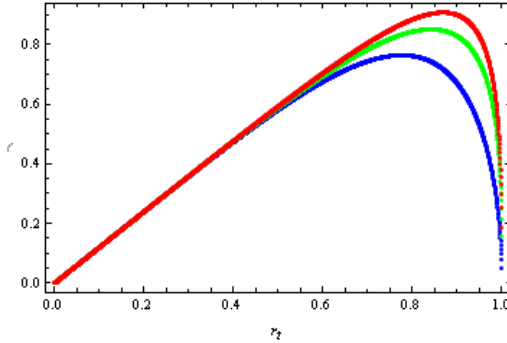


Figure 1: ℓ as a function of r_t for the cases of $Q = 0, 0.3, 0.5$ which are shown by red, green and, blue, respectively. The blue curve corresponds to the case of $r_H = r_\phi$ while the red one represents the neutral black brane.

goes to zero at both bounds and reaches a maximum value in this interval.

This behavior can be demonstrated by solving the integral (2.11) numerically. To do so, by making use of a scaling, without loss of generality, one may set $r_H = 1$. Then the only parameter of the model is the charge of the solution. Note that in this case one has $0 \leq Q^2 \leq \frac{1}{2d(d-1)}$. The neutral black brane corresponds to $Q = 0$, while $Q^2 = \frac{1}{2d(d-1)}$ is the case where $r_H = r_\phi$. The behavior of ℓ as a function of r_t for different values of Q for $d = 2$ are shown in the figure 1.

Form the equation (2.11) one may, in principle, find the turning point as a function of ℓ . Then plugging the result into the equation (2.12) we get an expression for Γ as a function ℓ . It is important to note that since ℓ is not a one-to-one function of r_t one has to make sure that the resultant Γ is minimum. Of course it is clear that the minimum Γ is obtained from the minimum r_t .

It should also be noticed that since the space time has a horizon one could always imagine the case where the function Γ is minimized by another hypersurface consisting of two disconnected parallel surfaces suspending between boundary and horizon. Therefore it is crucial to see which one is smaller.

The disconnected solution is given by setting $r_t = r_H$ in the expression of Γ by which we arrive at

$$\Gamma^{\text{diss}} = \frac{\tau L^{d-2} R^d}{G_{d+2}} \int_\epsilon^{r_H} dr \frac{\sqrt{1 - \phi^2}}{r^d}. \quad (2.14)$$

In general, depending on the parameters of the model either of connected or disconnected solutions could be smaller. In order to compare these two solutions it is useful to define the difference between them as follows

$$\Delta\Gamma = \Gamma^{\text{con}} - \Gamma^{\text{dis}} \quad (2.15)$$

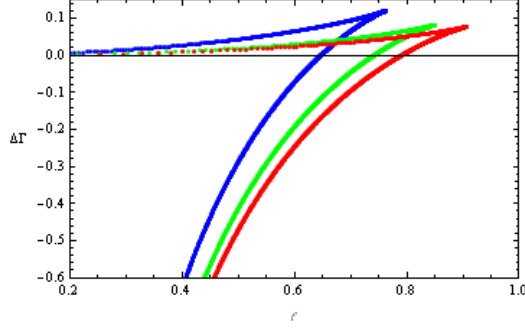


Figure 2: $\Delta\Gamma$ as a function of ℓ for the cases of $Q = 0, 0.3, 0.5$ which are shown by red, green and blue, respectively. The blue curve corresponds to the case where $r_H = r_\phi$ and blue one corresponds to the neutral case. Here we have set $r_H = 1$ and $\frac{\tau L^{d-2} R^d}{G_{d+2}} = 1$.

$$= \frac{\tau L^{d-2} R^d}{G_{d+2}} \left[\int_0^{r_t} dr \left(\frac{\sqrt{1-\phi^2}}{r^d \sqrt{1 - \left(\frac{r}{r_t}\right)^{2d} \frac{f_t}{f}}} - \frac{\sqrt{1-\phi^2}}{r^d} \right) - \int_{r_t}^{r_H} dr \frac{\sqrt{1-\phi^2}}{r^d} \right].$$

Note that, although, both connected and disconnected solutions are UV divergent, the UV contribution drops out in the difference leading to a finite number. The behaviors of $\Delta\Gamma$ as a function of ℓ for different values of Q for $d = 2$ case have been drawn in the figure 2.

One observes that for sufficiently small ℓ the closed hypersurface minimizes the expression of Γ , though there is a critical width over which the disconnected solution is favored. Moreover the critical width is always smaller than the maximum value the width can reach. Therefore one may conclude that the system exhibits a phase transition before it reaches the maximum ℓ . It is worth to note that as we increase the charge the maximum width becomes smaller and the phase transition occurs at smaller width, nevertheless as long as $Q \leq Q_c$ the behavior is universal.

Therefore as far as the qualitative behavior of Γ is concerned the effects of gauge field are not important and the main contributions come from the metric. In fact the background gauge field only affects the position of the horizon. We note, however, that as we increase the background charge one expects the effects of background charges become important as we shall explore in the following subsection.

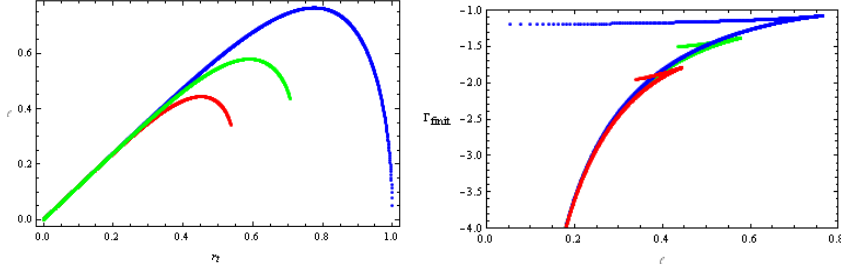


Figure 3: ℓ and Γ as functions of r_t and ℓ for $Q = 0.5, 1, \sqrt{3}$ which are shown by blue, green and red, respectively. Note that $Q = 0.5$ corresponds to the case of $r_H = r_\phi$ and we have plotted it just for a comparison. Here we have set $r_H = 1$ and $\frac{\tau L^{d-2} R^d}{G_{d+2}} = 1$.

2.2 $r_{\min} = r_\phi$ case

To study the effects of the background gauge field one may increase the background charge⁴ so that $Q > Q_c$ where $r_{\min} = r_\phi$. This indicates that the maximum value the turning point can get is r_ϕ . More precisely one has $0 \leq r_t \leq r_\phi < r_H$. In other words, since the turning point cannot reach the horizon we will not have the disconnected solution.

Indeed looking at the equation (2.11) one finds that although the width vanishes in the limit of $r_t \rightarrow 0$, it terminates at a non-zero value as one approaches r_ϕ . By making use of the numerical method the width ℓ can be found as a function of turning point which has been depicted in the figure 3 (left).

Moreover since in the present case we do not have the disconnected solution, it does not make sense to compute the difference $\Delta\Gamma$. Indeed the function Γ is the quantity we may want to compute. We note that due to the UV contribution, Γ diverges and has to be regulated by a UV cut off. More precisely one gets

$$\Gamma = \frac{\tau L^{d-2} R^d}{G_{d+2}} \int_\epsilon^{r_t} dr \frac{\sqrt{1-\phi^2}}{r^d \sqrt{1 - \left(\frac{r}{r_t}\right)^{2d} \frac{f_t}{f}}} = \frac{1}{G_{d+2}} \frac{\tau L^{d-2} R^d}{(d-1)\epsilon^{d-1}} + \Gamma_{\text{finite}}. \quad (2.16)$$

Subtracting the divergence part, it is then straightforward to calculate the finite part, Γ_{finite} , numerically. The results are shown in figure 3 (right).

From our numerical results one observes that as long as we are in the range of $Q_c < Q \leq \sqrt{(d+1)/(d-1)}$, qualitatively the behavior of Γ is universal, though it decreases as one increases the charge. Indeed, there is a critical width, ℓ_c above that both Γ and ℓ are not single valued functions. In other words for each width $\ell > \ell_c$

⁴ Since in our notations we have set $r_H = 1$ there is an upper bound on the background charge. More precisely the allowed values of background charge is $1/(2d(d-1)) \leq Q^2 \leq (d+1)/(d-1)$. Note that $Q^2 = (d+1)/(d-1)$ corresponds to the extremal case where $T = 0$.

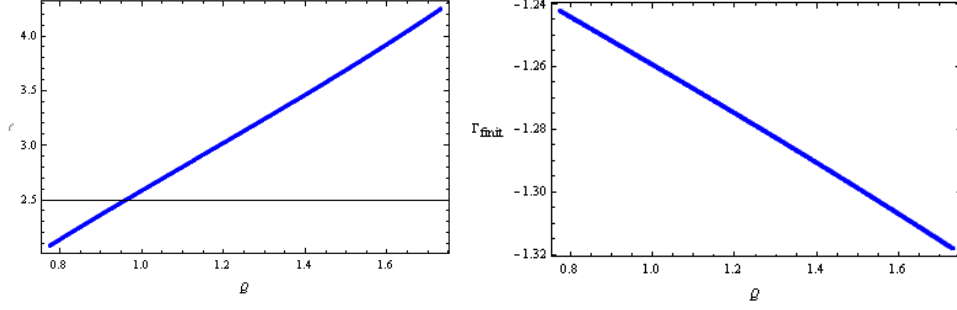


Figure 4: ℓ and Γ_{finite} as functions of Q for the case of $r_t = r_\phi$. The numerical values are for $r_H = 1$ and $R = L = 1$. Note that for all values of Q in the above plots we have $r_\phi < r_H$.

there are two turning points. Of course the favored Γ corresponds to the smaller turning point. Moreover there is a maximum width over which there is no a closed hypersurface. It is important to note that the width gets its maximum value before the turning point reaches its maximum value at r_ϕ .

An interesting observation we have made is as follows. Although there is a maximum width (or correspondingly a maximum turning point) over which there is no a closed hypersurface which minimizes Γ , there is a single closed hypersurface when $r_t = r_\phi$. Actually, as we have already mentioned, in this case r_t is not a turning point and indeed the hypersurface can cross the $r = r_\phi$ point and reaches the horizon. In fact it is easy to see that for this case the horizon is a turning point. Therefore we will get a single distinctive closed hypersurface which can probe the charged horizon while the effects of charges are important. In this case the corresponding expressions for ℓ and Γ are given by

$$\frac{\ell}{2} = \int_0^{r_H} dr \frac{\left(\frac{f_\phi}{f^2}\right)^{1/2} \left(\frac{r}{r_\phi}\right)^d \sqrt{1-\phi^2}}{\sqrt{1 - \left(\frac{r}{r_\phi}\right)^{2d} \frac{f_\phi}{f}}}, \quad \Gamma = \frac{\tau L^{d-2} R^d}{G_{d+2}} \int_\epsilon^{r_H} dr \frac{r^{-d} \sqrt{1-\phi^2}}{\sqrt{1 - \left(\frac{r}{r_\phi}\right)^{2d} \frac{f_\phi}{f}}}, \quad (2.17)$$

where $f_\phi = f(r_\phi)$. In the figure 4 we have depicted the behaviors of ℓ and finite part of Γ as functions of Q . Note that as we increase the background charges the width also increases linearly though the finite part of Γ decreases linearly.

3 Black hole in global AdS

In this section we extend our study to a charged black hole in a global AdS geometry. The action is still given by (2.1). The corresponding $d+2$ dimensional charged black

hole may be written as [13]

$$\begin{aligned} ds^2 &= \frac{R^2}{r^2} \left(-f(r) dt^2 + \frac{dr^2}{f(r)} + R^2 d\Omega_d^2 \right), \quad F_{rt} = -QR\sqrt{2d(d-1)}r^{d-2}, \\ f(r) &= 1 + \frac{r^2}{R^2} - \left(1 + \frac{r_+^2}{R^2} + Q^2 r_+^{2d} \right) \left(\frac{r}{r_+} \right)^{d+1} + Q^2 r^{2d}, \end{aligned} \quad (3.1)$$

where in our notation $d\Omega_d^2 = d\theta^2 + \cos^2 \theta d\Omega_{d-1}^2$ with $d\Omega_{d-1}^2$ being the metric of a $(d-1)$ -sphere, and r_+ is the location of the horizon which is a solution of $f(r) = 0$. We note that in general $f = 0$ has two real positive solutions and the horizon is given by the smallest root. The Hawking temperature and chemical potential are

$$T = \frac{d+1}{4\pi r_+} \left[1 - \frac{d-1}{d+1} \left(Q^2 r_+^{2d} - \frac{r_+^2}{R^2} \right) \right], \quad \mu = \sqrt{\frac{2d}{d-1}} QRr_+^{d-1}. \quad (3.2)$$

Using the corresponding Euclidean action the phase space of this system has been studied in [13] where it was shown that the theory has a rich phase space. Indeed the system could be thought of as either a grand canonical ensemble or a canonical ensemble depending on whether one wants to keep chemical potential or electric charge fixed, respectively. In either cases there are critical values for the parameters over which the model exhibits different behaviors.

Holographic geometric entropy in this background has also been studied [12] where it was shown that it may provide a useful order parameter to probe different phases of the system. Note that since in this case one, usually, performs a double Wick rotation there are two different ways to embed the hypersurface in the bulk. One could either consider $r(t)$ or $r(\theta)$. Actually by making use of these embedding it was observed in [12] that the resultant phase structures are very similar to that obtained from the Euclidean action [13]. We note, however, that since in what follows we are interested in the effects of the gauge field, as defined in the equation (1.2), the $r(t)$ embedding should automatically be excluded.

Therefore we will consider a subsystem in the form of $S^{d-2} \times \mathbb{R} \times I$, with I being an interval along θ direction given by $0 \leq \theta \leq 2\pi \frac{\ell}{R}$ with $\ell < R$. The extension of this subsystem to the bulk leads to a hypersurface whose profile is given by $\theta = \theta(r)$. Thus the induced metric on the hypersurface is

$$ds^2 = \frac{R^2}{r^2} \left[-f dt^2 + \left(\frac{1}{f} + R^2 \theta'^2 \right) dr^2 + R^2 \cos^2 \theta d\Omega_{d-2}^2 \right] \quad (3.3)$$

Therefore we arrive at

$$\Gamma = \frac{\tau V_{d-2} R^d}{G_{d+2}} \int_{\epsilon}^{r_t} dr \frac{\cos^{d-2} \theta}{r^d} \sqrt{1 - \phi^2 + f R^2 \theta'^2}, \quad (3.4)$$

where V_{d-2} is the volume of $(d-2)$ -sphere with radius R and r_t is the turning point

where $\theta'(r)$ diverges.

Alternatively, for $d \geq 3$ one may use a notation in which

$$d\Omega_d^2 = d\psi^2 + \cos^2 \psi d\theta^2 + \sin^2 \psi (d\phi^2 + \cos^2 \phi d\Omega_{d-3}^2), \quad (3.5)$$

and thus the corresponding subsystem may be chosen so that $\phi = \text{constant}$. The constant may be set to $\phi = \pi/2$ and the profile of the hypersurface is given by $\psi(r)$. Therefore the induced metric is

$$ds^2 = \frac{R^2}{r^2} \left[-f dt^2 + \left(\frac{1}{f} + R^2 \cos^2 \psi \theta'^2 \right) dr^2 + R^2 d\psi^2 + R^2 \sin^2 \psi d\Omega_{d-3}^2 \right]. \quad (3.6)$$

So that

$$\Gamma = \frac{\tau V_{d-3} R^d}{G_{d+2}} \int_{\epsilon}^{r_t} d\psi dr \frac{\cos^{d-3} \psi}{r^d} \sqrt{1 - \phi^2 + f R^2 \cos^2 \psi \theta'^2}. \quad (3.7)$$

Now the aim is to minimize Γ given in the equation (3.4) or (3.7), which can be done by treating them as actions for θ . In what follows we will mainly consider the first case where Γ is given by the equation (3.4) where unlike the previous cases, except for $d = 2$, the momentum conjugate of θ is not a constant of motion and therefore one needs to directly solve the equation of θ which is

$$\frac{d}{dr} \left(\frac{\cos^{d-2} \theta}{r^d} \frac{f R^2 \theta'}{\sqrt{1 - \phi^2 + f R^2 \theta'^2}} \right) + (d-2) \sin \theta \cos^{d-3} \theta \frac{\sqrt{1 - \phi^2 + f R^2 \theta'^2}}{r^d} = 0. \quad (3.8)$$

This equation may be solved with proper boundary conditions to find θ as a function of r_t . The corresponding boundary conditions could be $\theta(r \rightarrow 0) = 2\pi \frac{\ell}{R}$ and $\theta(r_t) = 0$. Then plugging the result into the equation (3.4) one can find Γ as a function of ℓ .

Although it is not explicitly clear from the above equation, there is still a special point at $r = r_\phi$ where $\phi = 1$ and the minimization makes sense for $r \leq r_\phi$. Indeed the situation is very similar to what we have considered in the previous section for the black brane. In particular for $r_+ \leq r_\phi$ the function Γ may also be minimized by a disconnected hypersurface which in the present case is given by

$$\Gamma^{\text{diss}} = \frac{V_{d-2} R^{d-1}}{G_{d+1}} \int_{\epsilon}^{r_H} dr \frac{\cos^{d-2} \theta_0}{r^d} \sqrt{1 - \phi^2}, \quad (3.9)$$

where $\theta_0 = \theta(r = 0)$. It is then natural to look for $\Delta\Gamma$ as a function of ℓ which might be thought of as an order parameter.

To proceed let us first consider $d = 2$ case in which the momentum conjugate of θ is, indeed, a constant of motion

$$\frac{R\theta'}{\sqrt{1 - \phi^2 + f R^2 \theta'^2}} = \left(\frac{r}{r_t} \right)^d \frac{f_t^{1/2}}{f}, \quad (3.10)$$

where r_t is the turning point. so that

$$\ell = \frac{1}{\pi} \int_0^{r_t} dr \frac{\left(\frac{f_t}{f^2}\right)^{1/2} \left(\frac{r}{r_t}\right)^2 \sqrt{1-\phi^2}}{\sqrt{1-\left(\frac{r}{r_t}\right)^4 \frac{f_t}{f}}}, \quad \Gamma = \frac{\tau R^2}{G_4} \int_\epsilon^{r_t} dr \frac{\sqrt{1-\phi^2}}{r^2 \sqrt{1-\left(\frac{r}{r_t}\right)^4 \frac{f_t}{f}}}, \quad (3.11)$$

which have essentially the same form as the corresponding expressions we have found in the previous section, though the function f is different. Therefore one expects that the system may exhibit the same behavior as in the black brane. In particular one can show that as long as we are in the range of the parameters where $r_+ \leq r_\phi$, the corresponding width, ℓ , vanishes at both $r_t = 0$ and $r_t = r_+$ points, while for $r_\phi < r_+$ although the width vanishes at r_t , it tends to a non-zero constant as $r_t \rightarrow r_\phi$. Moreover $r_t = r_\phi$ is not a turning point and the hypersurface can cross the point of $r = r_\phi$ to reach the horizon which is, indeed, the turning point in this case.

In order to calculate Γ one distinguishes two different cases depending on whether $r_+ \leq r_\phi$ or $r_+ > r_\phi$. Indeed for sufficiently small charges, *i.e.* $Q \leq Q_c$, where we are in the region of $r_+ \leq r_\phi$ the main contributions come from the metric and the effects of the charge is only due to the location of the horizon which is encoded in the metric's components. Indeed in this case the behavior of Γ is the same as the holographic geometric entropy.

On the other hand as one increases the background charge so that $Q > Q_c$ one reaches the region $r_\phi < r_+$ where the effects of the background charge become important. In this region since there is no place where the hypersurface can end, the minimization procedure does not lead to the disconnected solution.

It is worth to mention that for $d \geq 3$ using the expression of Γ given in the equation (3.7) we get exactly the same behavior as that in $d = 2$ discussed above which is, indeed, the same as what we have found in the previous section displayed in the figures 1 and 3.

On the other hand using the expression (3.4) for $d \geq 3$ although qualitatively we get the same behavior, a new feature appears when we change the ratio of R/r_+ . Of course as far as the effect of the gauge field is concerned the situation remains unchanged. Namely $\phi = 1$ sets a scale which controls the effects of the gauge field as before.

In order to explore the new feature let us consider the situation where $0 \leq Q^2 \leq \frac{1}{2d(d-1)}$ which corresponds to the case of $r_+ \leq r_\phi$. Note that in this region the effect of the background field is irrelevant and indeed we could have done the same for the geometric entropy. To proceed it is useful to study the behavior of $\Delta\Gamma$ which we will do that by using a numerical method. By making use of a scaling one may set $r_+ = 1$. It is important to note that unlike the black brane case where $\Delta\Gamma$ depends on R just through a trivial overall factor, in the present case it appears in the function f and therefore it may affect the behavior of the order parameter. To find the corresponding behavior numerically we will fix the dimension and the

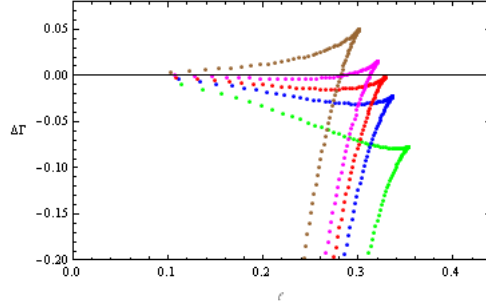


Figure 5: $\Delta\Gamma$ as a function of ℓ for $Q = 0.3, r_+ = 1$ and, $R = 0.6, 0.66, 0.69, 0.72, 0.8$ which are shown by green, blue, red, magenta and brown respectively. Note that as far as $0 \leq Q \leq 0.5$ where $r_+ \leq r_\phi$ one gets qualitatively the same behavior.

charge, and therefore we are left with a free parameter R which controls the behavior of the order parameter. Indeed one observes that for R of order of r_+ or bigger the model undergoes a phase transition though for a sufficiently small R/r_+ it exhibits no phase transition. More precisely there is a critical R/r_+ that indicates whether the system exhibits a phase transition. In the figure 5 we have summarized the above discussions by plotting $\Delta\Gamma$ as a function of ℓ for different values of R .

4 New proposal

In this section we would like to suggest a new quantity which enjoys the same properties as that of (1.2). To proceed consider a $d + 1$ dimensional subspace in a $d + 1$ dimensional field theory which has a gravity description. One may be interested in the information confined in this subsystem. Following our previous experiences in the holographic descriptions of non-local objects, it is natural to consider a hypersurface in the bulk geometry whose intersection with the boundary coincides to the $d + 1$ dimensional subsystem. For this hypersurface let us define the following quantity

$$C = \frac{1}{RG_{d+2}} \int d^{d+1}\xi \sqrt{-\det \left(\tilde{g} + R \sum_i^n F_{ab}^i \right)} \quad (4.1)$$

where

$$\tilde{g}_{ab} = \frac{\partial x^\mu}{\partial \xi_a} \frac{\partial x^\nu}{\partial \xi_b} g_{\mu\nu}, \quad F_{ab}^i = \frac{\partial x^\mu}{\partial \xi_a} \frac{\partial x^\nu}{\partial \xi_b} F_{\mu\nu}^i, \quad (4.2)$$

are the induced metric and the pull back of the gauge field on the world volume of the hypersurface, respectively. To compare this quantity with (1.2) it is illustrative to compute the above quantity for the black brane solution (2.2). Indeed for a

subsystem given by

$$0 \leq t \leq \tau, \quad -\frac{\ell}{2} \leq x_d \leq \frac{\ell}{2}, \quad 0 \leq x_i \leq L, \quad i = 1, \dots, d-1, \quad (4.3)$$

the quantity (4.1) reads

$$C = \frac{\tau L^{d-1} R^d}{G_{d+2}} \int dr \frac{\sqrt{1 - \phi^2 + f x'^2}}{r^{d+1}}, \quad (4.4)$$

where, as before, $\phi = \sqrt{2d(d-1)} Q r^d$. Following the procedure we have utilized in the previous sections one finds

$$\ell = 2 \int_0^{r_t} dr \left(\frac{r}{r_t} \right)^{d+1} \left(\frac{f_t}{f^2} \right)^{1/2} \frac{\sqrt{1 - \phi^2}}{\sqrt{1 - \left(\frac{r}{r_t} \right)^{2(d+1)} \frac{f_t}{f}}}, \quad (4.5)$$

and

$$C = \frac{\tau L^{d-1} R^d}{G_{d+2}} \int_\epsilon^{r_t} dr \frac{\sqrt{1 - \phi^2}}{r^{d+1} \sqrt{1 - \left(\frac{r}{r_t} \right)^{2(d+1)} \frac{f_t}{f}}}. \quad (4.6)$$

It is easy to demonstrate that these expressions exhibit the same behaviors as we have found from (1.2) in the previous sections. In particular there is a scale defined by setting $\phi = 1$ where the effects of the background gauge field become important. Nevertheless there are several advantages to work with (4.1).

First of all we note that in order to compute the geometric entropy or the one defined by the equations (1.2) one, generally, has to fix a *spatial* coordinate which in turn may lead to different results depending on which special direction is chosen. This is, specially, the case where we have spatial anisotropic scaling in the metric and a priori it is not obvious which one is favored. Actually we have already seen such a possibility in the previous section where we have studied black holes in the global AdS geometry. On the other hand in the present case where all coordinates are involved there is no such a confusion and as a result we find a unique expression.

Another advantage to work with the quantity (4.1) is the fact that, unlike the previous case, it is sensitive to both electric and magnetic charges. In order to explore the effects of the magnetic field, let us consider a four dimensional dyonic black brane whose solution may be written as

$$ds^2 = \frac{R^2}{r^2} \left(-f dt^2 + \frac{dr^2}{f} + dx^2 + dy^2 \right), \quad F_{rt} = -2RQ_e, \quad F_{xy} = 2RQ_m$$

$$f(r) = 1 - \left(1 + (Q_e^2 + Q_m^2) r_H^4 \right) \left(\frac{r}{r_H} \right)^3 + (Q_e^2 + Q_m^2) r^4. \quad (4.7)$$

For a strip subsystem given by

$$0 \leq t \leq \tau, \quad -\frac{\ell}{2} \leq x \leq \frac{\ell}{2}, \quad 0 \leq y \leq L, \quad (4.8)$$

the induced metric is

$$ds_{\text{ind}}^2 = \frac{R^2}{r^2} \left[-f dt^2 + \left(\frac{1}{f} + x'^2 \right) dr^2 + dy^2 \right]. \quad (4.9)$$

Therefore the equation (4.1) reads

$$C = \frac{\tau L R^2}{G_4} \int dr \frac{\sqrt{1 - \phi^2 + F x'^2}}{r^3}, \quad (4.10)$$

where $\phi = 2Q_e r^2$ and $F = (1 + 4Q_m^2 r^4)f$. We note that it has the same form as the one for electric case given by the equation (4.4) with just replacing f by F . Thus one arrives at

$$\ell = 2 \int_0^{r_t} dr \frac{\left(\frac{F_t}{F^2}\right)^{1/2} \left(\frac{r}{r_t}\right)^3 \sqrt{1 - \phi^2}}{\sqrt{1 - \left(\frac{r}{r_t}\right)^6 \frac{F_t}{F}}}, \quad C = \frac{\tau L R^2}{G_4} \int_\epsilon^{r_t} dr \frac{\sqrt{1 - \phi^2}}{r^3 \sqrt{1 - \left(\frac{r}{r_t}\right)^6 \frac{F_t}{F}}}. \quad (4.11)$$

Since we are interested in the effects of the magnetic charge, in what follows we will set $Q_e = 0$ which in turns results to dropping the factor of $\sqrt{1 - \phi^2}$ in the above expressions. In other words in the present case r_ϕ is always at infinity and, in our notation, $r_{\min} = r_H$.

On the other hand since the electric and magnetic charges appear symmetrically in the solution (4.7), one might naively expect that for generic values of magnetic charge the situation is the same as that in the electric case when $r_{\min} = r_H$. In this case the effects of the charge is negligible and effectively the results are the same as that if we had probed the background by (4.1) without the F_{ab} term.

We note, however, that as one increases the value of the magnetic charge we would expect to see a non-trivial effect. In particular for the case where the background becomes extremal, one should see a significant effect. Indeed this is the case as it has been shown in the figure 6.

It is interesting to see that even in the case where the electric charge is set to zero the width ℓ approaches a finite non zero value at $r = r_H$. Moreover the system exhibits a non-trivial phase transition even at the extremal limit. This has to be compared with the case that there is no a factor of F in the expression (4.1) where at the extremal limit the system exhibits no phase transition.

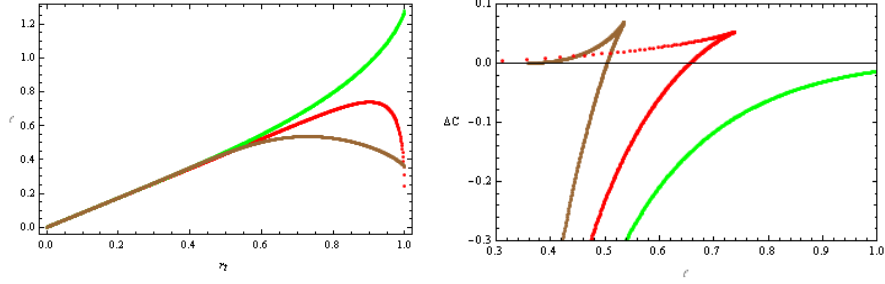


Figure 6: ℓ and $\Delta\Gamma$ as functions of r_t and ℓ , respectively. The brown curve corresponds to the case of $Q_e^2 = 0, Q_m^2 = 3$ where the dyonic black brane is extremal. This curve should be compared with the extremal solution without taking into account the effects of background gauge field (green), and the neutral solution where $Q_e = Q_m = 0$ (red). Here we have set $r_H = 1$ and $\frac{\tau LR^2}{G_4} = 1$.

5 Discussions

In this paper we have introduced a quantity which is sensitive to the background fractionalized charge not only due to its effects in the components of the metric, but also directly from the gauge field. To explore its properties we have explicitly computed the quantity for the RN black-brane and a black-hole in an asymptotically AdS geometry.

For sufficiently small charges the metric plays the essential roles, while as one increases the charge one would expect to see the effects of the gauge field. Indeed following our definition of the quantities (1.2) or (4.1), there is natural scale over which the direct effects of gauge field become significant. To elaborate this point it is illustrative to study the induced metric in more detail. To proceed it is useful to recall the following identity

$$\sqrt{-\det(\tilde{g} + RF_{ab})} = \left[\det(\tilde{g}) \det(G) \right]^{1/4}, \quad (5.1)$$

where

$$G_{\mu\nu} = \tilde{g} - R^2 F_{\mu\rho} \tilde{g}^{\rho\sigma} F_{\sigma\nu}. \quad (5.2)$$

In our case, using the explicit expression for x' obtained, for example, from the equation (2.8) the induced metric may be recast to the following form

$$ds_{\text{ind}}^2 = \frac{R^2}{r^2} \left[-f dt^2 + \left(\frac{f - f_t \phi^2 \left(\frac{r}{r_t} \right)^{2d}}{f - f_t \left(\frac{r}{r_t} \right)^{2d}} \right) \frac{dr^2}{f} + \sum_{i=1}^{d-2} dx_i^2 \right], \quad (5.3)$$

which shows that there is a horizon at $r = r_H$, as expected. On the other hand for

the metric $G_{\mu\nu}$ one finds

$$ds_{\text{open}}^2 = \frac{R^2}{r^2} \left\{ \frac{f(1-\phi^2)}{f-f_t\phi^2\left(\frac{r}{r_t}\right)^{2d}} \left[-f dt^2 + \left(\frac{f-f_t\phi^2\left(\frac{r}{r_t}\right)^{2d}}{f-f_t\left(\frac{r}{r_t}\right)^{2d}} \right) \frac{dr^2}{f} \right] + \sum_{i=1}^{d-2} dx_i^2 \right\}, \quad (5.4)$$

that indicates a possibility of having a natural scale at $r = r_\phi$ where $\phi = 1$, though the original geometry is smooth at this point. Indeed for $Q \leq Q_c$ one has $r_H \leq r_\phi$. Therefore the scale r_ϕ is behind the horizon and does not play an essential role indicating that the main contributions come from the metric. In fact in this case the effect of the charge is only through the components of the metric which in turn may fix the position of the horizon.

On the other hand in the opposite limit when $Q > Q_c$ where one has $r_\phi \leq r_H$ the effect of background gauge field is important and for a generic value of r_t the solution is well defined if $0 \leq r_t < r_\phi$. Note that in this case one gets a “bubble solution” and therefore the horizon cannot be probed. Of course for the special value of $r_t = r_\phi$ the metric (5.4) is well defined at $r = r_\phi$ and, indeed, it have a horizon at $r = r_H$.

Probably the most interesting, but rather difficult, aspect of our study, is to find an interpretation for the quantity defined by the equation (4.1) from dual field theory point of view. We should admit that we do not have a good understanding of this point, though using the results of the pure AdS case one may speculate the following relation with the geometric entropy. In fact for the pure AdS_{d+2} geometry one arrives at

$$C = \frac{\tau L^{d-1} R^d}{G_{d+2}} \left[\frac{1}{d\epsilon^d} - \frac{2^d \pi^{(d+1)/2}}{d} \left(\frac{\Gamma\left(\frac{d+2}{2(d+1)}\right)}{\Gamma\left(\frac{1}{2(d+1)}\right)} \right)^{d+1} \frac{1}{\ell^d} \right]. \quad (5.5)$$

Comparing this expression with the geometric entropy of the pure AdS geometry given by (2.13), up to a numerical factor, one finds

$$C \sim -L \frac{\partial \Gamma}{\partial \ell}, \quad (5.6)$$

which in turn may be related to the central charge of the theory [14]. It would be interesting to explore this point better.

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